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Quantum Logics and Completeness Criteria of Inner Product Spaces

Anatolij Dvurečenskij[1,2]

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We present the survey of measure-theoretic completeness criteria for inner product spaces using methods and notions important for quantum logics. Moreover, some new criteria and open problems are given.

1. INTRODUCTION

Let S be a real or complex inner product space with an inner product $(\cdot, \cdot)$. We recall that for $M \subseteq S$, $M \neq \emptyset$, by $M^\perp$ we mean the set of all $x \in S$ such that $(x, y) = 0$ for each $y \in M$. We introduce the following eight families of closed subspaces that show quite different properties from the ordering point of view:

1. $W(S)$ is the set of all closed subspaces of S which is a weakly orthocomplemented, complete lattice for which $M \vee M^\perp = S$, or $M = M^{\perp\perp}$, does not hold in general.
2. $F(S)$ is the set of all orthogonally closed subspaces of S, i.e., of all subspaces M of S such that $M = M^{\perp\perp}$, which is an orthocomplemented complete lattice (not necessarily orthomodular).
3. $D(S)$ is the set of all Foulis–Randall subspaces of S, i.e., of all subspaces M for which there exists an orthonormal system (ONS, for short) $\{u_i\}$ such that $M = \{u_i\}^{\perp\perp}$, which is a complete orthoposet. Any M of $D(S)$ possesses at least one local complement M', i.e., such an element $M' \in D(S)$ for which $M' \perp M$ and $M \vee M' = S$.

[1]Institute for Theoretical Physics, University of Cologne, D-5000 Cologne 41, Germany.
[2]Permanent address: Mathematical Institute, Slovak Academy of Sciences, CS-814 73 Bratislava, Czecho-Slovakia.

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4. $R(S)$ is the set of all subspaces M of S such that $M=\{u_i\}^{\perp\perp}$ for all maximal orthonormal systems (MONS, for short) $\{u_i\}$ of M, which is a poset.

5. $V(S)$ is the set of all subspaces M of S such that $M=\{u_i\}^{\perp\perp}$ and $M^{\perp}=\{v_j\}^{\perp\perp}$ for every MONS $\{u_i\}$ and $\{v_j\}$ of M and $M^{\perp}$, respectively, which is an orthocomplemented poset.

6. $E(S)$ is the set of all subspaces of S for which the condition $M+M^{\perp}=S$ holds, which is an orthomodular poset, and which is not necessarily a σ-poset.

Finally, we introduce:

7. $C(S)$ is the set of all subspaces of S of finite or cofinite dimension, which is an orthocomplemented, orthomodular lattice.

8. $P(S)$ is the set of all subspaces of S of finite dimension.

It is easy to see that

$$P(S)\subseteq C(S)\subseteq E(S)\subseteq V(S)\subseteq R(S)\subseteq D(S)\subseteq F(S)\subseteq W(S) \tag{1}$$

It is well known that S is complete iff $E(S)=W(S)$, or iff $E(S)=F(S)$, or iff $F(S)=W(S)$, and for an incomplete S proper inclusions in (1) are possible. All the above families play a considerable role in the axiomatic model of quantum mechanics (see, for example, Sherstnev, 1974). For quantum logic theory, among the most important notions are a measure or a charge (=signed measure) and a quantum logic. Hence, it is extremely important to find conditions on the above families to be quantum logics, and charges that characterize Hilbert spaces among inner product spaces.

A mapping m from $K(S)$, where K is the capital from the set $\{C, E, V, R, D, F, W\}$, into the real line R such that

$$m\left(\bigoplus_{i\in I} M_i\right)=\sum_{i\in I} m(M_i) \tag{2}$$

and for $K=W$ we add the condition

$$m(M\vee M^{\perp})=m(S), \qquad M\in W(S) \tag{3}$$

whenever $\{M_i\colon i\in I\}$ is a system of mutually orthogonal subspaces of $K(S)$ for which the join $\bigoplus_{i\in I} M_i$ exists in $K(S)$, is said to be a *charge*, *signed measure*, or *completely additive signed measure* if (2) holds for any finite, countable, or arbitrary index set I. If m attains only positive values, we say that m is a *finitely additive measure*, *measure*, or *completely additive measure*, respectively, according to the cardinality of I. A finitely additive measure m such that $m(S)=1$ is said to be a *state*. A charge is said to be *Jordan* if it can be represented as a difference of two positive, finitely additive measures.

If H is a Hilbert space over S, i.e., $H=\bar{S}$, Sherstnev's (1974) generalization of the famous Gleason (1957) theorem (see also Drisch, 1979; Dvurečenskij, 1988, 1989*a*) says that for any bounded signed measure m on the set of all closed subspaces of H, $L(H)$, of a separable Hilbert space H, dim $H\neq 2$, there exists a unique Hermitian operator T of trace class on H such that

$$m(M)=\mathrm{tr}(TP^M), \qquad M\in L(H) \tag{4}$$

where P^M is the orthoprojector from H onto M. Moreover, any Hermitian operator T of trace class on H generates a Jordan (bounded), completely additive signed measure m on $L(H)$ for any H.

Measure-theoretic criteria of the completeness of inner product spaces shall be divided into three groups: ones using (1) completely additive signed measures; (2) signed measures, and (3) charges, respectively.

2. COMPLETELY ADDITIVE SIGNED MEASURE CRITERIA

Theorem 1. An inner product space S is complete iff $K(S)$, where $K\in\{C, E, V, R, D, F, W\}$, possesses at least one nonzero completely additive signed measure.

Hamhalter and Pták (1987) showed that a separable inner product space S is complete iff $F(S)$ possesses at least one probability measure. This result has been generalized to $E(S)$, $F(S)$, and other families of subspaces for a general S, as well as to signed measures and finitely additive measures in series of papers (Dvurečenskij, 1989*a*; Dvurečenskij and Mišík, 1988; Dvurečenskij and Pulmannová, 1988, 1989; Dvurečenskij and Neubrunn, 1990, 1992).

A mapping $f\colon \mathscr{S}(S)=\{x\in S\colon \|x\|=1\}\to(-\infty, \infty)$ such that there is a constant W called the weight of f for which we have

$$\sum_i f(x_i)=W \tag{5}$$

for any MONS $\{x_i\}$ in S is said to be a frame function. Frame functions are in a one-to-one correspondence with completely additive signed measures on $K(S)$.

Theorem 2 (Dvurečenskij, 1989*b*). An inner product space S is complete iff S possesses at least one nonzero frame function.

Gudder (1974, 1975) and Gudder and Holland (1975) proved that S is complete iff any MONS of S is an ONB of S, i.e.,

$$\forall x\in S, \quad \forall\ \mathrm{MONS}\{x_i\} \text{ of } S, \qquad x=\sum_i (x_i, x)x_i$$

This result can be considerably improved by Theorem 2:

$$\exists 0 \neq x \in S(\bar{S}) \quad \forall \text{ MONS}\{x_i\} \text{ of } S, \qquad x = \sum_i (x_i, x) x_i$$

if we put $f(u) = |(u, x)|^2$ for any $u \in \mathscr{S}(S)$.

3. SIGNED MEASURE COMPLETENESS CRITERIA

A cardinal I is said to be nonmeasurable if on the power set 2^I there does not exist any probability measure vanishing on all one-point subsets of I.

Theorem 3 (Dvurečenskij, 1989*a*,*b*). An inner product space S is complete iff $K(S)$, if $K \in \{D, F, W\}$ and the dimension of S is a nonmeasurable cardinal, possesses at least one nonzero signed measure.

Theorem 4. S is complete iff $F(S)$ or $W(S)$ possesses at least one signed measure nonvanishing on $P(S)$.

Problem 1. Is S complete if the dimension of S is a nonmeasurable cardinal $>\aleph_0$ and $E(S)$ possesses at least one nonzero signed measure?

Problem 2. Is S complete if any splitting subspace M of S, $\dim S = \aleph_0$, is complete?

Now we present a new criterion, and for that, according to Cattaneo *et al.* (1987), we introduce the following notions. An ONS $\{u_t\}$ of S is Dacey iff $\{u_t\} \subseteq \{x\}^\perp \cup \{y\}^\perp$ implies $x \perp y$ for $x, y \in S$. It can be shown that ONS$\{u_t\}$ is Dacey iff $\{u_t\} = \{u_{t_i}\} \cup \{u_{t_j}\}$ and $\{u_{t_i}\} \cap \{u_{t_j}\} = \varnothing$, then $\{u_{t_i}\}^{\perp\perp} = \{u_{t_j}\}^\perp$.

Every Dacey ONS is a MONS. On the other hand, it is possible to find a MONS (Cattaneo and Marino, 1986) which is not Dacey. We say that an inner product space S is Dacey iff any MONS of S is Dacey. From Cattaneo *et al.* (1987) we have that S is Dacey iff $V(S) = D(S)$.

Theorem 5. A Dacey inner product space S is complete iff $K(S)$, where $K \in \{V, R, D\}$, possesses at least one signed measure nonvanishing on $P(S)$.

Proof. We know that S is Dacey iff $D(S) = V(S)$. Hence, $D(S) = R(S) = V(S)$. Suppose that $D(S)$ possesses at least one signed measure. Let $\{x_i\}$ be a countable ONS of S and put $M = \{x_i\}^{\perp\perp}$. Then $M \in D(S)$, and for any MONS$\{y_i\}$ of M we have, according to the basic property of $R(S)$, that $\{y_i\}^{\perp\perp} = M$. Without loss of generality we may assume that $m(\text{sp}(e)) \neq 0$ for at least one $e \in M$. Hence, f, where $f(x) = m(\text{sp}(x))$, $x \in \mathscr{S}(M)$, is a nonzero frame function on M with the weight $m(M)$. Using the criterion in Dvurečenskij (1989*b*), we conclude that M is complete and $M \in E(S)$, so that S is complete. QED

4. CHARGE CRITERIA

A charge m on $F(S)$ is said to be *$P(S)$-regular* if for each $M \in F(S)$ and each $\varepsilon > 0$ there exists a finite-dimensional subspace N of M such that $|m(M \cap N^{\perp})| < \varepsilon$. Let T be a Hermitian trace operator on $\bar{S}$; then a mapping m_T on $E(S)$ defined via

$$m_T(M) = \mathrm{tr}(TP^{\bar{M}}), \qquad M \in E(S) \tag{6}$$

is a $P(S)$-regular Jordan charge. In particular, if for any unit vector x of S we put

$$m_x(M) = \|x_M\|^2, \qquad M \in E(S)$$

if

$$x = x_M + x_{M^{\perp}}, \qquad x_M \in M, \quad x_{M^{\perp}} \in M^{\perp}$$

we obtain a system of $P(S)$-regular states on $E(S)$, $\{m_x : x \in \mathscr{S}(S)\}$, which determines, e.g., the ordering on $E(S)$.

Theorem 6. Any Jordan charge m on S, $\dim S \neq 2$, can be uniquely expressed as a sum $m = m_1 + m_2$, where m_1 is a $P(S)$-regular Jordan charge and m_2 is a Jordan charge vanishing on $P(S)$. A Jordan charge m is $P(S)$-regular iff m is of the form (6) for some Hermitian trace operator T on $\bar{S}$.

This result has a close connection with the decomposition of measures on quantum logics (Rüttimann, 1990).

We recall that a charge m is strongly $P(S)$-regular if, for every sequence $\{Q_n\}_n$ of mutually orthogonal elements of $K(S)$ such that $Q = \bigoplus_n Q_n$ exists in $K(S)$, there is a system of mutually compatible elements $\mathscr{B} \subset P(S)$ [i.e., $\mathscr{B}$ is a subset contained in a Boolean subalgebra of $K(S)$] such that, for each $\varepsilon > 0$ and every $R \in \{Q, Q_1^{\perp}, Q_2^{\perp}, \ldots\}$ there exists a $P \in \mathscr{B}$ with $P \subseteq R$ and $|m(R \cap P^{\perp})| < \varepsilon$. The strong $P(S)$-regularity implies the $P(S)$-regularity, but the converse is not true, in general.

Theorem 7 (Dvurečenskij *et al.*, 1990; Dvurečenskij, 1990*b*). If $\dim S = \aleph_0$, S is complete iff $K(S)$, if $K \in \{E, V, R\}$, possesses at least one nonzero, strongly $P(S)$-regular, finitely additive measure.

Problem 3. Does the strong $P(S)$-regularity of a nonzero Jordan charge on $E(S)$, $\dim S = \aleph_0$, imply the completeness of S?

Theorem 8 (Dvurečenskij, n.d.). S is complete iff $F(S)$ or $W(S)$ possesses at least one nonzero $P(S)$-regular Jordan charge.

We say that a subspace M_0 of S, $M_0 \in K(S)$, where $K \in \{F, W\}$, is a *support* of a finitely additive state m on $K(S)$ if $M(N) = 0$ iff $N \perp M_0$. If m

has a support, then it is unique. The support of m is $P(S)$-regular (with respect to m) if given $\varepsilon > 0$ there is a finite-dimensional subspace M of M_0 such that $m(M_0 \cap M^{\perp}) < \varepsilon$. For example, this is true if M_0 is of finite dimension.

Theorem 9. S is complete iff $K(S)$, where $K \in \{F, W\}$, possesses at least one finitely additive state with $P(S)$-regular support. Moreover, this state is a completely additive state.

Proof. According to Dvurečenskij (1991), m is expressible in the form $m(M) = \mathrm{tr}(TP^{\bar{M}}) + m_2(M)$, $M \in F(S)$, where T is a positive Hermitian operator of trace class on $\bar{S}$, and m_2 is a positive function vanishing on $P(S)$. Due to our assumptions, there is a sequence of nondecreasing subspaces of the support M_0, $\{M_n\}$, such that $m(M_0) = \lim_n m(M_n)$. Therefore, $m(M_0^{\perp}) = 0 = m_2(M_0^{\perp})$ and

$$\begin{aligned}\mathrm{tr}(TP^{\overline{M_0}}) + m_2(M_0) = m(M_0) = \lim_n m(M_n) \\ = \lim_n \mathrm{tr}(TP^{M_n}) + \lim_n m(M_n) = \lim_n \mathrm{tr}(TP^{M_n})\end{aligned}$$

which gives

$$m(M) = \mathrm{tr}(TP^{\bar{M}}) \qquad \text{for any} \quad M \in F(S)$$

In other words, m is a $P(S)$-regular state, so that, in view of the previous theorem, S is complete, and m is completely additive.

For $K = W$ we proceed in an analogous way to that in the previous paragraph. QED

Problem 4. Is the set of all (bounded) nonzero charges on $K(S)$, where $K \in \{V, R, D, F, W\}$ for incomplete S, nonempty? What is its connection to the completeness of S?

Problem 5. If $K \in \{D, F, W\}$, $\dim S \neq 2$, then (Dvurečenskij, 1990*a*) $K(S)$ does not possess any two-valued state. Is this true for $K \in \{E, V, R\}$?

Problem 6. Can any Jordan charge vanishing on $P(S)$ be extended to a Jordan charge on $L(\bar{S})$?

The notion of a finitely additive measure can be extended in a conventional way to a measure attaining the values $+\infty$, too. A finitely additive measure m on $E(S)$ is said to be $P(S)_\infty$-regular if, given $M \in E(S)$, there is a nondecreasing sequence of finite-dimensional subspaces of M, $\{M_n\}$, such that $m(M) = \lim_n m(M_n)$. For finite, finitely additive measures, $P(S)$-regularity and $P(S)_\infty$-regularity coincide.

Dvurečenskij (1992) describes the set of all $P(S)_\infty$-regular, finitely additive measures (which are, for example, σ-finite, i.e., m is σ-finite if there is a countable, orthogonal decomposition $\{M_n\}$ of splitting subspaces of S such that $\bigoplus_{n=1}^{\infty} M_n = S$ and $m(M_n) < \infty$ for any n) for the case that S is complete and dim $S \neq 2$.

Problem 7. Describe the set of all (σ-finite) $P(S)_\infty$-regular, finitely additive measures on $E(S)$ for incomplete S.

We review the following completeness criteria.

Theorem 10. Let S be an inner product space. The following statements are equivalent:

1. S is complete.
2. $E(S) = W(S)$ (Gudder, 1974).
3. $F(S) = W(S)$ (Gudder, 1974).
4. If M is a proper closed subspaces of S, then $M^\perp \neq \{O\}$ (Gudder, 1974).
5. If f is a continuous linear functional on S, there exists $y \in S$ such that $f(x) = (x, y)$ for all $x \in S$ (Gudder, 1974).
6. Every MONS in S is an ONB in S (Gudder and Holland, 1975).
7. $F(S)$ is orthomodular (Amemiya and Araki, 1966/1967).
8. $E(S) = F(S)$.
9. $E(S)$ is a complete lattice (Gross and Keller, 1977).
10. $E(S)$ is a σ-lattice (Cattaneo and Marino, 1986).
11. $E(S)$ is a σ-orthoposet (=quantum logic) (Dvurečenskij, 1988).
12. $E(S)$ possesses the join of any sequence of mutually orthogonal one-dimensional subspaces of S (Dvurečenskij, 1988).
13. $R(S) = F(S)$ (Cattaneo *et al.*, 1987).
14. $D(S) = E(S)$ (Canetti and Marino, 1988).
15. $K(S)$, if $K \in \{C, E, V, R, D, F, W\}$, possesses at least one nonzero, completely additive signed measure (Dvurečenskij and Pulmannová, 1988, 1989).
16. S possesses at least one nonzero frame function (Dvurečenskij, 1989*b*, 1990*a*).
17. There exists a unit vector $y \in \bar{S}$ such that $y = \sum_i (y, x_i) x_i$ for any MONS $\{x_i\}$ in S (Dvurečenskij, 1989*b*, 1990*a*).
18. $F(S)$ $[W(S)]$ possesses at least one Jordan $P(S)$-regular, nonzero charge (Dvurečenskij, n.d.).
19. $K(S)$, where $K \in \{E, V, R\}$ and the dimension of S is a countable cardinal, possesses at least one nontrivial, strongly $P(S)$-regular, finitely additive measure (Dvurečenskij *et al.*, 1990; Dvurečenskij, 1990*b*).
20. $D(S)$, if S has dimension a nonmeasurable cardinal, possesses at least one nonzero, strongly $P(S)$-regular, finitely additive measure (Dvurečenskij *et al.*, 1990).

21. $K(S)$, where $K \in \{D, F, W\}$ and the dimension of S is a nonmeasurable cardinal, possesses at least one σ-additive, nontrivial signed measure (Dvurečenskij, 1989*a*, 1990*a*).

22. $K(S)$, where $K \in \{F, W\}$, possesses at least one signed measure nonvanishing on $P(S)$ (Dvurečenskij, n.d.).

23. $K(S)$, where $K \in \{F, W\}$ and S is Dacey, possesses at least one signed measure nonvanishing on $P(S)$ (Theorem 5).

24. $K(S)$, where $K \in \{F, W\}$, possesses at least one finitely additive state with a finite-dimensional support (Dvurečenskij, 1991).

25. $K(S)$, where $K \in \{F, W\}$, possesses at least one finitely additive state with a $P(S)$-regular support (Theorem 9).

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